COMPACT DIFFERENCES OF COMPOSITION OPERATORS ON HOLOMORPHIC FUNCTION SPACES IN THE UNIT BALL

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ABSTRACT. We find a lower bound for the essential norm of the difference of two composition operators acting on $H^2(B_N)$ or $A_s^2(B_N)$ $(s > -1)$. This result plays an important role in proving a necessary and sufficient condition for the difference of linear fractional composition operators to be compact, which answers a question posed by MacCluer and Weir in 2005.

1 Introduction

Let B_N denote the open unit ball in $\mathbb{C}^N$, with D for the unit disc B_1. We write σ to denote the normalized Lebesgue measure on the unit sphere ∂B_N, the Hardy space $H^2(B_N)$ is the set of all holomorphic functions f in B_N such that

$$||f||_2^2 := \sup_{0<r<1} \int_{\partial B_N} |f(r\zeta)|^2 d\sigma(\zeta) < \infty.$$

For $s > -1$, the standard weighted Bergman space $A_s^2(B_N)$ consists of holomorphic functions f in B_N satisfying

$$||f||_{2,s}^2 := \int_{B_N} |f(z)|^2 w_s(z) d\nu(z) < \infty,$$

where

$$w_s(z) = \frac{\Gamma(N+s+1)}{\Gamma(N+1)\Gamma(s+1)}(1-|z|^2)^s$$

and ν denotes the normalized Lebesgue volume measure on B_N. Write $H^2(B_N) = A_{-1}^2(B_N)$, it is well known that $A_s^2(B_N)$ is a Hilbert space of holomorphic functions in B_N with the reproducing kernel $K_z(w) = (1- < w, z >)^{-(N+s+1)}$ for any $s \geq -1$ (see [16] and [20]).

2000 Mathematics Subject Classification. Primary: 47B38; Secondary 32A35, 32A36.

Key words and phrases. Composition operators, Hardy space, Bergman spaces, compact differences.

*Liangying Jiang is supported by Shanghai Education Research and Innovation Project (No.10YZ185) and by Shanghai University Research Special Foundation for Outstanding Young Teachers (No.sjr09015)

†Caiheng Ouyang is supported by the National Natural Science Foundation of China (No.10971219)

We consider the composition operator C_φ acting on the hardy space $H^2(B_N)$ or the Bergman spaces $A_s^2(B_N)$ $(s > -1)$, defined by $C_\varphi f = f \circ \varphi$, where φ is an analytic map from B_N into B_N. When $N = 1$, the Littlewood Subordination Theorem shows that C_φ is bounded for any analytic self-map φ of D, and many other properties of C_φ have been characterized, see the good monographs [17] and [3] for details. However, for $N \geq 2$, one may find examples of $\varphi : B_N \to B_N$ such that C_φ is not bounded (see Section 3.5 in [3]). Moreover, some basic properties of composition operators in the setting of the ball are not easily managed. The purpose of this paper is to characterize those pairs φ and ψ for which the difference $C_\varphi - C_\psi$ is compact acting on $H^2(B_N)$ or $A_s^2(B_N)$ $(s > -1)$. From these results, one may derive some information about the structure of the space of composition operators.

The topological structure of the set of composition operators on $H^2(D)$ was first studied by Berkson in [1]. Shapiro and Sundberg [19] improved the result of Berkson and raised the problem about compact differences of composition operators. In [19] they found a lower bound for the essential norm $||C_\varphi - C_\psi||_e$ in terms of the measure of the set $E_\varphi = \{\zeta \in \partial D : |\varphi(\zeta)| = 1\}$, where φ and ψ are analytic self-maps of D and $\varphi(\zeta) := \lim_{r\to 1} \varphi(r\zeta)$. This result has been extended to the case of Hardy spaces $H^p(B_N)$ $(0 < p \leq \infty)$ (see [7] and [6]). On the other hand, using the angular derivative, MacCluer [12] discussed the differences of composition operators and obtained the following essential norm estimate

$$||C_\varphi - C_\psi||_e^2 \geq |\varphi'(\zeta)|^{-\beta}$$

with $\beta = 1$ for $H^2(D)$ and $\beta = s + 2$ for $A_s^2(D)$ $(s > -1)$, where $\varphi'(\zeta)$ is the angular derivative of φ at $\zeta \in \partial D$. Thus, from the results of Shapiro and Sundberg [19] and MacCluer [12], one may determine for which pairs φ and ψ the difference $C_\varphi - C_\psi$ is compact. Recently, Aleksandrov-Clark measures also have been used to study the compactness of differences and linear combinations of composition operators on the spaces mentioned (see [5], [11], [18]).

In Section 2 of this paper, we would expect similar results about the compact differences of composition operators on $H^2(D)$ and $A_s^2(D)$ $(s > -1)$ to hold in several variables. First, motivated by the work of MacCluer [12], we will find a lower bound for the essential norm of composition operator difference $C_\varphi - C_\psi$ on $H^2(B_N)$ or $A_s^2(B_N)$ $(s > -1)$, but with

$$d_\varphi(\zeta) = \liminf_{z\to\zeta} \frac{1 - |\varphi(z)|}{1 - |z|}$$

instead of $|\varphi'(\zeta)|$ (see Theorem 2.1 in Section 2). In fact, the Julia-Carathéodory theorem in the disc shows that if φ has finite angular derivative at $\zeta \in \partial D$ then $|\varphi'(\zeta)| = d_\varphi(\zeta)$. So the Julia-Carathéodory theory in B_N (see [16] or [3]) will be a key tool for its proof. Note that in the proof of MacCluer's result (Theorem 2.2 of [12]), the main idea is to use

$$\lim_{z\to\zeta} \frac{1 - |\varphi(z)|}{1 - |z|} = |\varphi'(\zeta)|$$

as z approaches $\zeta \in \partial D$ nontangentially, which is a result of the Julia-Carathéodory Theorem in the disc (see [3]). However, for higher dimensions, we have not found the corresponding result and some techniques will be needed. Moreover, our method can be generalized to estimate essential norms of linear combinations of composition operators. As a consequence, we obtain some necessary conditions for differences or linear combinations of composition operators to be compact on $H^2(B_N)$ or $A_s^2(B_N)$ $(s > -1)$.

In another direction, Bourdon [2] treated the question on compact differences of linear fractional composition operators and proved that $C_\varphi - C_\psi$ is compact on $H^2(D)$ if and only if both C_φ and C_ψ are compact or $\varphi = \psi$. This result also holds on $A_s^2(D)$ $(s > -1)$ from Moorhouse's result [14]. For the linear fractional self-map φ of B_N with a boundary fixed point, MacCluer and Weir [13] showed that the difference $C_{\varphi\circ\sigma} - C_{\sigma\circ\varphi}$ is compact on $H^2(B_N)$ or $A_s^2(B_N)$ $(s > -1)$ if and only if $\varphi \circ \sigma = \sigma \circ \varphi$, where σ is the adjoint map of φ, and asked the following question:

$(*)$ *For distinct linear fractional self-maps φ and ψ of B_N can $C_\varphi - C_\psi$ ever be compact?*

In Section 3, we then focus on compact differences of linear fractional composition operators. For linear fractional self-maps φ and ψ of B_N, we will prove that $C_\varphi - C_\psi$ is compact on $H^2(B_N)$ or $A_s^2(B_N)$ $(s > -1)$ if and only if both C_φ and C_ψ are compact or $\varphi = \psi$ (Theorem 3.1 in Section 3), which answers the question $(*)$. The basic ideas come from Bourdon [2] and MacCluer and Weir [13]. In our proof an important tool is the result about compact difference of composition operators in Section 2, which says that if $C_\varphi - C_\psi$ is compact then $\varphi(\zeta) = \psi(\zeta)$ and $d_\varphi(\zeta) = d_\psi(\zeta)$ at some point $\zeta \in \partial B_N$ (see Corollary 2.2 in Section 2). This will give a very useful information for the relations of matrixes associated with φ and ψ. In this point, our approach is different from that used by MacCluer and Weir [13].

For the proof of MacCluer and Weir's result [13], under the condition of φ having a boundary fixed point, assume e_1 to be fixed, they obtained $\varphi \circ \sigma(e_1) = \sigma \circ \varphi(e_1)$. They also found that the adjoint maps of $\varphi \circ \sigma$ and $\sigma \circ \varphi$ are themselves and then deduced that $D_1(\varphi \circ \sigma)_1(e_1) = D_1(\sigma \circ \varphi)_1(e_1) = 1$. According to the proof of Lemma 4.2 in [10], we see that this result always holds in the case of φ fixing e_1. Note that the Julia-Carathéodory Theorem in B_N gives $D_1(\varphi \circ \sigma)_1(e_1) = d_{\varphi\circ\sigma}(e_1)$ and $D_1(\sigma \circ \varphi)_1(e_1) = d_{\sigma\circ\varphi}(e_1)$. Hence, if φ fixes a boundary point, MacCluer and Weir in fact obtained the same result as ours, that is $\varphi \circ \sigma(e_1) = \sigma \circ \varphi(e_1)$ and $d_{\varphi\circ\sigma}(e_1) = d_{\sigma\circ\varphi}(e_1)$. However, if $||\varphi||_\infty = 1$, this will hold automatically from the compactness of $C_{\varphi\circ\sigma} - C_{\sigma\circ\varphi}$ by Corollary 2.2 in Section 2. Thus, the hypothesis that φ fixes a boundary point in their result can be replaced by a weaker condition $||\varphi||_\infty = 1$ (see Theorem 3.2 in Section 3).

This work is part of the first author's doctoral thesis (see [9]), but, at that time, the method for proving $\gamma_k = \gamma'_k$ $(k = 1, \cdots, n)$ in the proof of Theorem 3.1 was not correct. In this paper, we improve the proof of Theorem 3.1 and obtain some other results. Recently, the authors learned that Heller et al [8] independently proved Theorem 3.1 using different methods.

2 Essential norms of composition operator differences and linear combinations

The essential norm of an operator T on the space $\mathcal{H}$ is defined by $||T||_e = \inf\{||T - K|| : K \text{ is compact on } \mathcal{H}\}$. In [12], MacCluer considered the topological space of composition operators and obtained the following result.

Theorem A. *Let $\varphi, \psi : D \to D$ be analytic maps and suppose that φ has a finite angular derivative at $\zeta \in \partial D$. Consider C_φ and C_ψ acting on $H^2(D)$ or $A^2_s(D)$ for $s > -1$. Then unless $\psi(\zeta) = \varphi(\zeta)$ and $\psi'(\zeta) = \varphi'(\zeta)$, one have*

$$||C_\varphi - C_\psi||_e^2 \geq |\varphi'(\zeta)|^{-\beta},$$

where $\beta = 1$ for the space $H^2(D)$ and $\beta = s + 2$ for the spaces $A^2_s(D)$.

If φ and ψ have radial limits of modulus 1 at $\zeta \in \partial D$ with $\varphi(\zeta) = \psi(\zeta)$ and $|\varphi'(\zeta)| = |\psi'(\zeta)|$, we say that φ and ψ have the same data at this point (see [12]). Immediately, from Theorem A, one may get that if $C_\varphi - C_\psi$ is compact then φ and ψ must have the same data for those points, at which φ has finite angular derivatives.

In this section, we will discuss the analogue of Theorem A for the ball, but in higher dimensions, our lower bound needs a corresponding form of the angular derivative $|\varphi'(\zeta)|$ of $\varphi : D \to D$. First, we summarize some relevant results on the angular derivative and the Julia-Carathéodory theory in the ball.

A curve Γ in B_N will be called a $\zeta-$ curve if Γ approaches a point $\zeta \in \partial B_N$. We say that a function $f : B_N \to \mathbb{C}$ has restricted limit L at $\zeta \in \partial B_N$, if $\lim_{t\to 1} f(\Gamma(t)) = L$ for every $\zeta-$ curve $\Gamma(t)$ that satisfies

$$\lim_{t\to 1} \frac{|\Gamma(t) - \gamma(t)|^2}{1 - |\gamma(t)|^2} = 0$$

and

$$\frac{|\gamma(t) - \zeta|}{1 - |\gamma(t)|} \leq M < \infty \qquad \text{for } 0 \leq t < 1,$$

where $\gamma(t) =< \Gamma(t), \zeta > \zeta$ is the projection of Γ onto the complex line through ζ. In this case, the curve Γ is said to be restricted and its orthogonal projection γ is nontangential (see [16]).

Let φ be an analytic self-map of B_N and $\zeta \in \partial B_N$, if there exists a point $\eta \in \partial B_N$ such that the restricted limit of

$$\frac{< \eta - \varphi(z), \eta >}{< \zeta - z, \zeta >}$$

exists then φ is said to have finite angular derivative at ζ. By the Julia-Carathéodory Theorem in B_N, this is equivalent to

$$d_\varphi(\zeta) = \liminf_{z\to\zeta} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty,$$

where z approaches ζ unrestrictedly in B_N. Moreover, under these conditions, φ has restricted limit η at ζ and $D_\zeta \varphi_\eta(z) =< \varphi'(z)\zeta, \eta >$ has restricted limit $d_\varphi(\zeta)$.

Next, making use of the Julia-Carathéodory Theorem in B_N, we will give lower bounds for essential norms of differences and linear combinations of composition operators on $H^2(B_N)$ or $A^2_s(B_N)$ $(s > -1)$. Therefore, some information about the compactness of them can be obtained.

Theorem 2.1. *Let φ and ψ be analytic self-maps of B_N. Suppose that they induce bounded composition operators on $H^2(B_N)$ or $A^2_s(B_N)$ $(s > -1)$ and φ has finite angular derivative at $\zeta \in \partial B_N$. Then, unless $\psi(\zeta) = \varphi(\zeta)$ (as radial limits) and $d_\psi(\zeta) = d_\varphi(\zeta)$, we have*

$$||C_\varphi - C_\psi||_e^2 \geq d_\varphi(\zeta)^{-\beta},$$

where $\beta = N$ for $H^2(B_N)$ and $\beta = N + s + 1$ for $A^2_s(B_N)$.

Proof. If φ has finite angular derivative at $\zeta \in \partial B_N$, by the Julia-Carathéodory Theorem in B_N, there exists a point $\eta \in \partial B_N$ such that $\varphi(\zeta) := \lim_{r\to 1} \varphi(r\zeta) = \eta$. Assume that U and V are unitary transformations on B_N which send e_1 to ζ and η respectively, where $e_1 = (1, 0, \ldots, 0) = (1, 0')$. Let V^* be the adjoint of V with $V^* = V^{-1}$. Then the map $\phi(z) = V^*\varphi U(z)$ also has finite angular derivative at e_1 and

$$\begin{aligned} d_\phi(e_1) &= \lim_{r\to 1} D_1\phi_1(re_1) = \lim_{r\to 1} < \phi'(re_1)e_1, e_1 >= \lim_{r\to 1} < V^*\varphi'(U(re_1))Ue_1, e_1 > \\ &= \lim_{r\to 1} < \varphi'(r\zeta)\zeta, \eta >= \lim_{r\to 1} D_\zeta\varphi_\eta(r\zeta) = d_\varphi(\zeta), \end{aligned}$$

where ϕ_1 denotes the first coordinate function of ϕ. Moreover, write $\tau = V^*\psi U$, we have

$$||C_\phi - C_\tau||_e = ||C_{V^*\varphi U} - C_{V^*\psi U}||_e = ||C_U(C_\varphi - C_\psi)C_{V^*}||_e = ||C_\varphi - C_\psi||_e.$$

So the proof will be complete if we can show that the result holds for ϕ and τ. Thus, we may assume $\zeta = \eta = e_1$.

Let K_z be the reproducing kernel for $z \in B_N$, it is easy to see

$$||C_\varphi - C_\psi||_e^2 = ||(C_\varphi - C_\psi)^*||_e^2 \geq \limsup_{|z|\to 1} \frac{||(C_\varphi - C_\psi)^*K_z||^2}{||K_z||^2}.$$

Since $C_\varphi^* K_z = K_{\varphi(z)}$, we can write

$$\begin{aligned} \frac{||(C_\varphi - C_\psi)^*K_z||^2}{||K_z||^2} &= \frac{||K_{\varphi(z)} - K_{\psi(z)}||^2}{||K_z||^2} \\ &= (1 - |z|^2)^\beta [||K_{\varphi(z)}||^2 + ||K_{\psi(z)}||^2 - 2\mathrm{Re}\, K_{\varphi(z)}(\psi(z))] \\ &= \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2}\right)^\beta + \left(\frac{1 - |z|^2}{1 - |\psi(z)|^2}\right)^\beta - 2\mathrm{Re}\left(\frac{1 - |z|^2}{1 - < \psi(z), \varphi(z) >}\right)^\beta, \end{aligned}$$

where the norm $||\cdot||$ is in the space $H^2(B_N)$ or the spaces $A_s^2(B_N)$ $(s>-1)$, and $K_z(w)=(1-<w,z>)^{-\beta}$ is the corresponding reproducing kernel (see Section 1).

Our goal is to estimate the first term and the third term on the last line of the equation above. If $\psi(e_1):=\lim\limits_{r\to 1}\psi(re_1)\neq e_1$, then there exists a sequence $\{r_n\}$ going to 1 as $n\to\infty$ such that $\lim\limits_{n\to\infty}\psi(r_ne_1)=w\neq e_1$, which implies

$$\lim_{n\to\infty}\mathrm{Re}\left(\frac{1-r_n^2}{1-<\psi(r_ne_1),\varphi(r_ne_1)>}\right)^{\beta}=0.$$

On the other hand, the proof of the Julia-Carathéodory Theorem in B_N gives

$$\lim_{n\to\infty}\frac{1-|\varphi(r_ne_1)|}{1-r_n}=d_\varphi(e_1).$$

Thus, we deduce that $||C_\varphi-C_\psi||_e^2\geq d_\varphi(e_1)^{-\beta}$.

Next, if $\lim\limits_{r\to 1}\psi(re_1)=e_1$ but $d_\psi(e_1)\neq d_\varphi(e_1)$, to deal with this case, the argument used to prove Theorem A is not helpful. For a self-map φ of D, if φ has finite angular derivative at $\zeta\in\partial D$, then

$$\lim_{z\to\zeta}\frac{1-|\varphi(z)|}{1-|z|}$$

has nontangential limit $|\varphi'(\zeta)|$ by the Julia-Carathéodory theory in the disc. However, we don't know whether this would happen in the setting of the ball and we need some different approaches. For any curve γ approaching 1 nontangentially in D, the curve $\Gamma\equiv\{z=(\lambda,0'):\lambda\in\gamma\}$ is a restricted e_1- curve in B_N. Note that the angular derivative of φ existing implies that $(1-\varphi_1(z))/(1-z_1)$ has restricted limit $d_\varphi(e_1)$ at e_1, so it tends to $d_\varphi(e_1)$ as z approaches e_1 along Γ. Define $\rho(\lambda)=\varphi_1(\lambda e_1)=\varphi_1(\lambda,0')$ for $\lambda\in D$, the above discussion shows that

$$\frac{1-\rho(\lambda)}{1-\lambda}=\frac{1-\varphi_1(\lambda,0')}{1-\lambda}$$

has finite nontangential limit $d_\varphi(e_1)$ as $\lambda\to 1$. Therefore, the map ρ has finite angular derivative at 1. By the Julia-Carathéodory Theorem in D, we see that

$$\lim_{\lambda\to 1}\frac{1-|\rho(\lambda)|}{1-|\lambda|}=d_\varphi(e_1)$$

as λ approaches 1 nontangentially. Combining

$$\frac{1-|\varphi(z_1,0')|}{1-|z_1|}\leq\frac{1-|\varphi_1(z_1,0')|}{1-|z_1|}=\frac{1-|\rho(z_1)|}{1-|z_1|}$$

with $\liminf\limits_{z\to e_1}\frac{1-|\varphi(z)|}{1-|z|}=d_\varphi(e_1)$, we get

$$\lim_{z_1\to 1}\frac{1-|\varphi(z_1,0')|}{1-|z_1|}=d_\varphi(e_1)$$

as z_1 tends to 1 nontangentially.

Now, we discuss two cases for $d_\psi(e_1) \neq d_\varphi(e_1)$. First, if $d_\psi(e_1) < \infty$, we compute that

$$\begin{aligned}\frac{1-<\psi(z),\varphi(z)>}{1-|z|^2} &= \frac{1-|\varphi(z)|^2}{1-|z|^2} + \frac{<\varphi(z)-\psi(z),\varphi(z)>}{1-|z|^2} \\ &= \frac{1-|\varphi(z)|^2}{1-|z|^2} + \frac{1-z_1}{1-|z|^2}\Bigg[\Big(\frac{1-\psi_1(z)}{1-z_1} - \frac{1-\varphi_1(z)}{1-z_1}\Big)\overline{\varphi_1(z)} \\ &\quad + \sum_{j=2}^N \Big(\frac{|\varphi_j(z)|^2}{1-z_1} - \frac{\psi_j(z)\overline{\varphi_j(z)}}{1-z_1}\Big)\Bigg].\end{aligned}$$

Let $\Gamma_{e_1,M} \equiv \{z=(z_1,0') \in B_N : \frac{|1-z_1|}{1-|z_1|^2} = M\}$. It is clear that $\Gamma_{e_1,M}$ is a restricted e_1- curve and the orthogonal projection of $\Gamma_{e_1,M}$ is nontangengtial. As z approaches e_1 along $\Gamma_{e_1,M}$, the previous argument shows that

$$\lim_{z\to e_1} \frac{1-|\varphi(z)|^2}{1-|z|^2} = d_\varphi(e_1).$$

On the other hand, by the Julia-Carathéodory theory in B_N, we have

$$\frac{1-\varphi_1(z)}{1-z_1} \to d_\varphi(e_1), \qquad \frac{1-\psi_1(z)}{1-z_1} \to d_\psi(e_1),$$

$$\frac{\varphi_j(z)}{(1-z_1)^{1/2}} \to 0, \qquad \frac{\psi_j(z)}{(1-z_1)^{1/2}} \to 0 \qquad \text{for } \ 2 \le j \le N$$

and $\varphi_1(z)$ has finite limit 1 as $z \to e_1$ along $\Gamma_{e_1,M}$. Note that $|(1-\overline{z_1})/(1-z_1)| = 1$, so

$$\frac{\overline{\varphi_j(z)}}{(1-z_1)^{1/2}} = \overline{\Big(\frac{\varphi_j(z)}{(1-z_1)^{1/2}}\Big)} \cdot \Big(\frac{1-\overline{z_1}}{1-z_1}\Big)^{1/2} \to 0$$

holds for $2 \le j \le N$. Write $1-z_1 = |1-z_1|e^{i\theta}$, all these results yield that

$$\lim_{\substack{z\in\Gamma_{e_1,M}\\ z\to e_1}} \frac{1-<\psi(z),\varphi(z)>}{1-|z|^2} = d_\varphi(e_1) + Me^{i\theta}(d_\psi(e_1)-d_\varphi(e_1)).$$

It follows that

$$\lim_{\substack{z\in\Gamma_{e_1,M}\\ z\to e_1}} \mathrm{Re}\Big(\frac{1-|z|^2}{1-<\psi(z),\varphi(z)>}\Big)^\beta$$

converges to 0 as M tends to infinity. Consequently, we obtain $||C_\varphi - C_\psi||_e^2 \ge d_\varphi(e_1)^{-\beta}$.

If $d_\psi(e_1) = \infty$, we use the following inequality (see [13])

$$\frac{||(C_\varphi - C_\psi)^* K_z||^2}{||K_z||^2} \ge \Big(\frac{||K_{\varphi(z)}|| - ||K_{\psi(z)}||}{||K_z||}\Big)^2 + 2(1-u(z))\frac{||K_{\varphi(z)}||}{||K_z||} \cdot \frac{||K_{\psi(z)}||}{||K_z||},$$

where $u(z) = (1-\rho^2(z))^{\beta/2}$ and $\rho(z)$ is the pseudohyperbolic distance between $\varphi(z)$ and $\psi(z)$ with

$$1-\rho^2(z) = \frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)}{|1-<\varphi(z),\psi(z)>|^2}.$$

Note that the second term on the right side of the inequality is not less than 0 and

$$\lim_{r\to 1}\frac{||K_{\varphi(re_1)}||}{||K_{re_1}||} = \lim_{r\to 1}\left(\frac{1-r^2}{1-|\varphi(re_1)|^2}\right)^{\frac{\beta}{2}} = d_\varphi(e_1)^{-\frac{\beta}{2}}.$$

At the same time, $d_\psi(e_1) = \liminf_{z\to e_1}\frac{1-|\psi(z)|^2}{1-|z|^2} = \infty$ implies

$$\lim_{r\to 1}\frac{||K_{\psi(re_1)}||}{||K_{re_1}||} = \lim_{r\to 1}\left(\frac{1-r^2}{1-|\psi(re_1)|^2}\right)^{\frac{\beta}{2}} = 0.$$

Thus, we have

$$||C_\varphi - C_\psi||_e^2 \geq \lim_{r\to 1}\frac{||(C_\varphi - C_\psi)^*K_{re_1}||^2}{||K_{re_1}||^2} \geq \lim_{r\to 1}\left(\frac{||K_{\varphi(re_1)}|| - ||K_{\psi(re_1)}||}{||K_{re_1}||}\right)^2 = d_\varphi(e_1)^{-\beta}$$

as desired. □

As a corollary of Theorem 2.1, we get a necessary condition for the difference $C_\varphi - C_\psi$ to be compact. This result will provide some heuristics for the proof of our theorem in Section 3.

Corollary 2.2. *Suppose that C_φ and C_ψ are bounded on $H^2(B_N)$ or $A_s^2(B_N)$ $(s > -1)$. If $C_\varphi - C_\psi$ is compact then $\varphi(\zeta) = \psi(\zeta)$ and $d_\varphi(\zeta) = d_\psi(\zeta)$ must hold at the point $\zeta \in \partial B_N$, where the angular derivative of φ exists.*

In fact, for the case $d_\psi(e_1) = \infty$ in the proof of Theorem 2.1, using similar idea of Kriete and Moorhouse in [11], we have

$$|K_{\varphi(z)}(\psi(z))| = |<K_{\varphi(z)}, K_{\psi(z)}>| \leq ||K_{\varphi(z)}||^{1/2}||K_{\psi(z)}||^{1/2}$$

from the Schwarz inequality. That is

$$\left(\frac{1-|z|^2}{|1-<\psi(z),\varphi(z)>|}\right)^\beta \leq \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\beta/2}\left(\frac{1-|z|^2}{1-|\psi(z)|^2}\right)^{\beta/2}.$$

In the proof of Theorem 2.1, we obtain

$$\lim_{\substack{z\in\Gamma_{e_1,M}\\ z\to e_1}}\frac{1-|z|^2}{1-|\varphi(z)|^2} = \frac{1}{d_\varphi(e_1)}.$$

Now, $(1-|z|^2)/(1-|\psi(z)|^2)$ tends to 0 as $z\to e_1$ unrestrictedly. Hence, in the case $d_\psi(e_1) = \infty$,

$$\lim_{\substack{z\in\Gamma_{e_1,M}\\ z\to e_1}}\frac{1-|z|^2}{|1-<\psi(z),\varphi(z)>|} = 0$$

holds and we also get that $||C_\varphi - C_\psi||_e^2 \geq d_\varphi(e_1)^{-\beta}$. Now, combining the above discussion with the proof of Theorem 2.1, for analytic self-maps φ and ψ of B_N with $\varphi(e_1) = e_1$, we deduce that

$$\lim_{M\to\infty} \lim_{\substack{z\in\Gamma_{e_1,M}\\ z\to e_1}} \frac{1-|z|^2}{1-<\psi(z),\varphi(z)>} = \begin{cases} \frac{1}{d_\varphi(e_1)}, & \text{if } \psi(e_1)=\varphi(e_1) \\ & \text{and } d_\psi(e_1)=d_\varphi(e_1)<\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we have the following result for linear combinations of composition operators.

Theorem 2.3. *Let $\phi_1, \ldots, \phi_m$ be a class of analytic self-maps of B_N, which induce bounded composition operators on $H^2(B_N)$ or $A_s^2(B_N)$ $(s > -1)$. Then for any complex numbers $c_1, \ldots, c_m$ and $\zeta \in \partial B_N$,*

$$||c_1\phi_1 + \cdots + c_m\phi_m||_e^2 \geq \sum_{d_{\phi_j}(\zeta)<\infty} \Bigg| \sum_{\substack{\phi_l(\zeta)=\phi_j(\zeta)\\ d_{\phi_l}(\zeta)=d_{\phi_j}(\zeta)}} c_l \Bigg|^2 \frac{1}{d_{\phi_j}(\zeta)^\beta}$$

with $\beta = N$ for $H^2(B_N)$ and $\beta = N + s + 1$ for $A_s^2(B_N)$.

Proof. As in the proof of Theorem 2.1, we may assume $\zeta = e_1$. It is clear that

$$\begin{aligned} ||c_1\phi_1 + \cdots + c_m\phi_m||_e^2 &\geq \limsup_{|z|\to 1} \left|\left| \Big(c_1\phi_1 + \cdots + c_m\phi_m\Big)^* \frac{K_z}{||K_z||} \right|\right|^2 \\ &\geq \lim_{M\to\infty} \lim_{\substack{z\in\Gamma_{e_1,M}\\ z\to e_1}} \left|\left| \Big(\overline{c_1}\phi_1^* + \cdots + \overline{c_m}\phi_m^*\Big) \frac{K_z}{||K_z||} \right|\right|^2 \\ &= \sum_{j,l=1}^m \overline{c_j}\, c_l \lim_{M\to\infty} \lim_{\substack{z\in\Gamma_{e_1,M}\\ z\to e_1}} \left(\frac{1-|z|^2}{1-<\phi_l(z),\phi_j(z)>} \right)^\beta . \end{aligned}$$

If ϕ_j has finite angular derivative at e_1 and $\phi_j(e_1) \neq e_1$ for some j, there exists a unitary transformation W such that $W\phi_j(e_1) = e_1$, then

$$\frac{1-|z|^2}{1-<W\phi_l(z),W\phi_j(z)>} = \frac{1-|z|^2}{1-<\phi_l(z),\phi_j(z)>}$$

and $d_{W\phi_j}(e_1) = d_{\phi_j}(e_1)$. Since the result proceeding Theorem 2.3 holds for the map $W\phi_j$, we then obtain that

$$\lim_{M\to\infty} \lim_{\substack{z\in\Gamma_{e_1,M}\\ z\to e_1}} \frac{1-|z|^2}{1-<\phi_l(z),\phi_j(z)>} = \begin{cases} \frac{1}{d_{\phi_j}(e_1)}, & \text{if } \phi_l(e_1)=\phi_j(e_1) \\ & \text{and } d_{\phi_l}(e_1)=d_{\phi_j}(e_1)<\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned}||c_1\phi_1+\cdots+c_m\phi_m||_e^2 &\geq \sum_{j=1}^{m}\sum_{l=1}^{m}\overline{c_j}\,c_l \lim_{M\to\infty}\lim_{\substack{z\in\Gamma_{e_1,M}\\ z\to e_1}}\left(\frac{1-|z|^2}{1-<\phi_l(z),\phi_j(z)>}\right)^{\beta}\\ &= \sum_{j=1}^{m}\Bigg(\sum_{\substack{\phi_l(e_1)=\phi_j(e_1)\\ d_{\phi_l}(e_1)=d_{\phi_j}(e_1)<\infty}}\overline{c_j}\,c_l\frac{1}{d_{\phi_j}(e_1)^{\beta}}\Bigg)\\ &= \sum_{d_{\phi_j}(e_1)<\infty}\Bigg|\sum_{\substack{\phi_l(e_1)=\phi_j(e_1)\\ d_{\phi_l}(e_1)=d_{\phi_j}(e_1)}}c_l\Bigg|^2\frac{1}{d_{\phi_j}(e_1)^{\beta}},\end{aligned}$$

which is the desired conclusion. $\square$

Immediately, we have the following result for the compactness of linear fractional combinations of composition operators.

Corollary 2.4. *Suppose that $C_{\phi_1},\ldots,C_{\phi_m}$ are bounded and $c_1\phi_1+\cdots+c_m\phi_m$ is compact when acting on $H^2(B_N)$ or $A_s^2(B_N)$ $(s>-1)$. For any point $\zeta\in\partial B_N$ at which if ϕ_j has finite angular derivative for some $j=1,\ldots,m$, then*

$$\sum_{\substack{\phi_l(\zeta)=\phi_j(\zeta)\\ d_{\phi_l}(\zeta)=d_{\phi_j}(\zeta)}} c_l=0.$$

3 Compact differences of linear fractional composition operators

A linear fractional map of $\mathbb{C}^N$ is defined by

$$\varphi(z)=\frac{Az+B}{<z,C>+d},$$

where $A=(a_{jk})$ is an $N\times N$ matrix, $B=(b_j)$, $C=(c_j)$ are $N\times 1$ column vectors, and d is a complex number. The matrix

$$m_\varphi=\begin{pmatrix} A & B\\ C^* & d\end{pmatrix}$$

is called a matrix associated with φ and we write $\varphi\sim m_\varphi$. If φ is a linear fractional map from B_N into B_N, Cowen and MacCluer [4] proved that the adjoint map σ given by

$$\sigma(z)=\frac{A^*z-C}{<z,-B>+\overline{d}}$$

maps B_N into itself. In [4] they also deduced that the composition operator C_φ is bounded on $H^2(B_N)$ and $A_s^2(B_N)$ $(s>-1)$.

In [13], MacCluer and Weir considered the essential normality of linear fractional composition operators on $H^2(B_N)$ or $A^2_s(B_N)$ $(s > -1)$, and obtained the following result concerning compact difference of two special composition operators. About compact differences of more general linear fractional composition operators, they raised the question $(*)$.

Theorem B. *Suppose φ is a linear fractional self-map of B_N with a boundary fixed point. The operator $C_{\varphi\circ\sigma} - C_{\sigma\circ\varphi}$ is compact on $H^2(B_N)$ or $A^2_s(B_N)$ $(s > -1)$ if and only if $\varphi \circ \sigma = \sigma \circ \varphi$.*

In this section, for linear fractional self-maps φ and ψ of B_N, we will give a necessary and sufficient condition for $C_\varphi - C_\psi$ to be compact on $H^2(B_N)$ or $A^2_s(B_N)$ $(s > -1)$, which completely answers the question on compact differences of linear fractional composition operators in several variables. In the proof of our result, Corollary 2.2 in Section 2 is an essential tool. On the other hand, in order to eventually deduce that the symbols of two composition operators are equivalent, no matter what case in the disc or in the ball, Bourdon [2] and MacCluer and Weir [13] all used the fact that if $C_\varphi - C_\psi$ is compact then

$$\rho(z_n)\frac{1 - |z_n|^2}{1 - |\varphi(z_n)|^2}$$

converges to zero for any sequence $\{z_n\}$ with $|z_n| \to 1$. This result will unavoidably be used in our proof and so some of our treatments may be similar to those in the proof of Theorem B.

Theorem 3.1. *Suppose that φ and ψ are linear fractional self-maps of B_N. The difference $C_\varphi - C_\psi$ is compact on $H^2(B_N)$ or $A^2_s(B_N)$ $(s > -1)$ if and only if either both C_φ and C_ψ are compact or $\varphi = \psi$.*

Proof. The sufficient condition is trivial, so we only need to prove the necessity. If $C_\varphi - C_\psi$ is compact, it is easy to see that C_φ and C_ψ must be compact or not at the same time. Now, we assume that C_φ is not compact, then there exist ζ and η on ∂B_N such that $\varphi(\zeta) = \eta$ (here, we have used the fact that C_φ is compact if and only if $||\varphi||_\infty < 1$ for linear fractional self-map φ of B_N). It follows that φ has finite angular derivative at ζ from the smoothness of φ on $\overline{B_N}$. Write $t = d_\varphi(\zeta)$, thus $0 < t < \infty$ by the Julia-Carathéodory Theorem in B_N. Moreover, applying Corollary 2.2, we see that $\psi(\zeta) = \varphi(\zeta)$ and $d_\psi(\zeta) = d_\varphi(\zeta)$ must hold. Similar to the proof of Theorem 2.1, it suffices to assume that $\zeta = \eta = e_1$.

First, we give some information for the matrix

$$m_\varphi = \begin{pmatrix} A & B \\ C^* & d \end{pmatrix}$$

associated with φ, where $A = (a_{jk})$, $B = (b_j)$ and $C = (c_j)$ for $j, k = 1, \ldots, N$, and $d > 0$. Since $\varphi(e_1) = e_1$, this gives

$$a_{11} + b_1 = \overline{c_1} + d \tag{3.1}$$

and $a_{j1}+b_j=0$ for $2\le j\le N$. Note that Lemma 6.6 of [3] shows that $D_j\varphi_1(e_1)=0$ and by Equation (3.1), we compute that $D_j\varphi_1(e_1)=(a_{1j}-\overline{c_j})/(\overline{c_1}+d)$ for $j=2,\dots,N$. Thus $a_{1j}=\overline{c_j}$ for $j=2,\dots,N$. On the other hand, by the Julia-Carathéodory Theorem in B_N, we have $t=d_\varphi(e_1)=D_1\varphi_1(e_1)=(a_{11}-\overline{c_1})/(\overline{c_1}+d)$. The arguments above yield that

$$m_\varphi=\begin{pmatrix} a_{11} & \overline{c_2} & \dots & \overline{c_N} & \overline{c_1}+d-a_{11}\\ -b_2 & & & & b_2\\ \vdots & & & & \vdots\\ -b_N & & & & b_N\\ \overline{c_1} & \overline{c_2} & \dots & \overline{c_N} & d \end{pmatrix}.$$

Since $\overline{c_1}+d\neq 0$, setting $K=\overline{c_1}/(\overline{c_1}+d)$, $\beta_j=b_j/(\overline{c_1}+d)$ and $\gamma_j=\overline{c_j}/(\overline{c_1}+d)$ for $2\le j\le N$, we obtain an equivalent matrix for φ up to multiply all entries of m_φ by the value $(\overline{c_1}+d)^{-1}$, that is

$$\varphi\sim T\equiv\begin{pmatrix} t+K & \gamma_2 & \dots & \gamma_N & 1-t-K\\ -\beta_2 & & & & \beta_2\\ \vdots & & & & \vdots\\ -\beta_N & & & & \beta_N\\ K & \gamma_2 & \dots & \gamma_N & 1-K \end{pmatrix}.$$

Note that we have proved that $\psi(e_1)=\varphi(e_1)=e_1$ and $d_\psi(e_1)=d_\varphi(e_1)=t$. Using similar discussions as above, we can get a matrix S for ψ with parameters K', β_j' and γ_j' replacing K, β_j and γ_j. For needed later, we denote the (j,k) entries of T and S by α_{jk} and α_{jk}' respectively, where $j,k=2,\dots,N$.

First, we will prove $K=K'$ and the argument is similar to Step 4 in the proof of Theorem B. For the convenience of the reader, we will give a proof in the case of the Hardy space. Define maps ρ, $\rho':D\to D$ by

$$\rho(\lambda)=\varphi_1(\lambda e_1)=\frac{(t+K)\lambda+1-t-K}{K\lambda+1-K},$$

$$\rho'(\lambda)=\psi_1(\lambda e_1)=\frac{(t+K')\lambda+1-t-K'}{K'\lambda+1-K'}.$$

If $K\neq K'$, it is clear that ρ and ρ' are distinct linear fractional self-maps of D with $\rho(1)=\rho'(1)=1$, then the difference $C_{\rho_j}-C_{\rho_j'}$ is not compact on $A^2_{N-2}(D)$ (see [14]). Hence, there exists a bounded sequence $\{f_n\}$ in $A^2_{N-2}(D)$ which tends to 0 uniformly on compact subsets of D, and

$$||(C_{\rho_j}-C_{\rho_j'})f_n||_{A^2_{N-2}(D)}\nrightarrow 0$$

as $n\to\infty$. Define functions $F_n(z_1,z')=f_n(z_1)$ for $(z_1,z')\in B_N$, we see that $\{F_n\}$ is a sequence tending to 0 uniformly on compact subsets of B_N and $||F_n||_{H^2(B_N)}=||f_n||_{A^2_{N-2}(D)}$ (see 1.4.5 in [16]). Setting $g_n(\lambda)=F_n\circ\varphi(\lambda e_1)-F_n\circ\psi(\lambda e_1)$ for $\lambda\in D$, by Proposition 2.21 in [3] this defines a restriction operator satisfying

$$||F_n\circ\varphi-F_n\circ\psi||_{H^2(B_N)}\ge||g_n||_{A^2_{N-2}(D)}.$$

Obviously, we have $g_n(\lambda) = f_n \circ \varphi_1(\lambda e_j) - f_n \circ \psi_1(\lambda e_j) = f_n \circ \rho(\lambda) - f_n \circ \rho'(\lambda)$. Therefore,

$$||F_n \circ \varphi - F_n \circ \psi||_{H^2(B_N)} \geq ||f_n \circ \rho_j - f_n \circ \rho'_j||_{A^2_{N-2}(D)}$$

and so $||(C_\varphi - C_\psi)F_n||_{H^2(B_N)}$ does not converge to 0 as $n \to \infty$. This contradicts with the fact that $C_\varphi - C_\psi$ is compact and so $K = K'$ holds.

Now, for $2 \leq k \leq N$ and $2 \leq j \leq N$, a computation shows that

$$D_1\varphi_1(e_1) = t, D_k\varphi_1(e_1) = 0,$$

$$D_{11}\varphi_1(e_1) = -2tK, D_{1k}\varphi_1(e_1) = -t\gamma_k, D_{kk}\varphi_1(e_1) = 0,$$

and

$$D_1\varphi_j(e_1) = -\beta_j, D_k\varphi_j(e_1) = \alpha_{jk},$$

$$D_{11}\varphi_j(e_1) = 2K\beta_j, D_{1k}\varphi_j(e_1) = -K\alpha_{jk} + \gamma_k\beta_j, D_{kk}\varphi_j(e_1) = -2\gamma_k\alpha_{jk},$$

where φ_j denotes the j^{th} component of φ. Since φ is holomorphic in a neighborhood of e_1, we have the following expansions:

$$\varphi_1(z_1, 0') = 1 + t(z_1 - 1) - tK(z_1 - 1)^2 + o(|z_1 - 1|^2) \tag{3.2}$$

and

$$\varphi_j(z_1, 0') = -\beta_j(z_1 - 1) + K\beta_j(z_1 - 1)^2 + o(|z_1 - 1|^2) \qquad \text{as } (z_1, 0') \to e_1. \tag{3.3}$$

Let $\Gamma \equiv \{z = (z_1, 0') \in B_N : 1 - |z_1|^2 = |1 - z_1|^2\}$ be a e_1- curve. It is clear that $\mathrm{Re}\, z_1 = |z_1|^2$ for any $z \in \Gamma$. Thus, for points $z = (z_1, 0') \in \Gamma$, we use (3.2) and (3.3) to obtain

$$\begin{aligned} 1 - |\varphi(z)|^2 &= -2t\mathrm{Re}\,(z_1 - 1) - t^2|z_1 - 1|^2 + 2t\mathrm{Re}\,[K(z_1 - 1)^2] \\ &\quad -|z_1 - 1|^2 \sum_{j=2}^{N} |\beta_j|^2 + o(|z_1 - 1|^2) \\ &= (2t - t^2)|z_1 - 1|^2 + 2t\mathrm{Re}\,[K(z_1 - 1)^2] - |z_1 - 1|^2 \sum_{j=2}^{N} |\beta_j|^2 + o(|z_1 - 1|^2). \end{aligned}$$

Note that $(z_1 - 1)^2/|z_1 - 1|^2 \to -1$ as z approaches e_1 along Γ, so $(1 - |z|^2)/(1 - |\varphi(z)|^2)$ has a finite limit

$$\frac{1}{2t - t^2 - 2t\mathrm{Re}\, K - \sum_{j=2}^{N} |\beta_j|^2}.$$

However, combining the compactness of $C_\varphi - C_\psi$ with Theorem 3 in [13], we know that

$$\rho(z)\frac{1 - |z|^2}{1 - |\varphi(z)|^2}$$

must tend to 0 as z approaches the boundary of B_N. This forces that $\rho(z)$ goes to 0 as z approaches e_1 along Γ, so

$$1 - \rho^2(z) = \frac{(1 - |\varphi(z)|^2)(1 - |\psi(z)|^2)}{|1 - < \varphi(z), \psi(z) > |^2}$$

converges to 1.

Using $K' = K$ and similar expansions as (3.2) and (3.3) for the components of ψ with β_j replaced by β'_j, we calculate that

$$\frac{1-|\psi(z)|^2}{1-|z|^2} \to 2t - t^2 - 2t\mathrm{Re}\, K - \sum_{j=2}^{N} |\beta'_j|^2$$

and

$$\begin{aligned} 1- < \varphi, \psi > &= 1 - |\psi|^2 - < \varphi - \psi, \psi > \\ &= 1 - |\psi|^2 - |z_1 - 1|^2 \sum_{j=2}^{N} \overline{\beta'_j}(\beta_j - \beta'_j) + o(|z_1 - 1|^2) \end{aligned}$$

as $z \to e_1$ along the curve Γ. Write $a = t - t^2 - 2t\mathrm{Re}\, K$, because

$$t = \liminf_{z\to e_1} \frac{1-|\varphi(z)|^2}{1-|z|^2} = \liminf_{z\to e_1} \frac{1-|\psi(z)|^2}{1-|z|^2}$$

as z approaches e_1 unrestrictedly, this implies that $a - \sum_{j=2}^{N} |\beta_j|^2 \geq 0$ and $a - \sum_{j=2}^{N} |\beta'_j|^2 \geq 0$. Therefore, as z tends to e_1 along Γ, $1 - \rho^2(z)$ converges to

$$\frac{(t + a - \sum_{j=2}^{N} |\beta_j|^2)(t + a - \sum_{j=2}^{N} |\beta'_j|^2)}{|t + a - \sum_{j=2}^{N} \beta_j \overline{\beta'_j}|^2}. \tag{3.4}$$

Set $L_1 = (\sum_{j=2}^{N} |\beta_j|^2)^{1/2}$, $L_2 = (\sum_{j=2}^{N} |\beta'_j|^2)^{1/2}$ and $I = \sum_{j=2}^{N} \beta_j \overline{\beta'_j}$. Since $|I| \leq L_1 L_2$, one may write $|I| = \theta L_1 L_2$ for $\theta \in [0, 1]$. The above discussion gives that the value in (3.4) is equal to 1, it follows that

$$\begin{aligned} 0 = \left| t + a - \sum_{j=2}^{N} \beta_j \overline{\beta'_j} \right|^2 - \left(t + a - \sum_{j=2}^{N} |\beta_j|^2 \right) \left(t + a - \sum_{j=2}^{N} |\beta'_j|^2 \right) \\ = (t + a)(L_1^2 + L_2^2 - 2\mathrm{Re}\, I) + |I|^2 - L_1^2 L_2^2 \\ \geq (t + a)(L_1^2 + L_2^2 - 2|I|) + |I|^2 - L_1^2 L_2^2 \\ = (t + a)(L_1^2 + L_2^2 - 2\theta L_1 L_2) + (\theta^2 - 1) L_1^2 L_2^2. \end{aligned} \tag{3.5}$$

Let $h(\theta) = (t + a)(L_1^2 + L_2^2 - 2\theta L_1 L_2) + (\theta^2 - 1)L_1^2 L_2^2$ for fixed L_1 and L_2. Observe that the derivative $h'(\theta) = -2(t + a)L_1 L_2 + 2\theta L_1^2 L_2^2 = 2L_1 L_2(\theta L_1 L_2 - a - t) < 0$ from $a \geq L_1^2$ and $a \geq L_2^2$. Thus, the function $h(\theta)$ is decreasing on $[0, 1]$ and $0 \geq h(\theta) \geq h(1)$ for any $\theta \in [0, 1]$. However, $h(1) = (t + a)(L_1 - L_2)^2 \geq 0$, this gives $h(1) = 0$ and then $L_1 = L_2$. Hence, the equalities in (3.5) and $h(\theta) \geq h(1)$ must be attained, which implies that $\mathrm{Re}\, I = |I|$ and $\theta = 1$. All these force that $\beta_j = \beta'_j$ for $2 \leq j \leq N$.

Next, we will prove that $\alpha_{jk} = \alpha'_{jk}$ for $j, k = 2, \ldots, N$. Fix $k \geq 2$, let $\Gamma_k(r) = re_1 + \sqrt{1 - r}e_k$ for $0 < r < 1$. As $r \to 1^-$, the following expansions hold,

$$\varphi_1(re_1 + \sqrt{1 - r}e_k) = 1 + t(r - 1) + o(1 - r)$$

and for $j \geq 2$,

$$\varphi_j(re_1 + \sqrt{1-r}e_k) = -\beta_j(r-1) + \alpha_{jk}\sqrt{1-r} - \gamma_k\alpha_{jk}(1-r) + o(1-r).$$

Since $\beta_j = \beta_j'$ for $2 \leq j \leq N$, we can obtain analogous expansions for the coordinates of ψ with parameters γ_k, α_{jk} replaced by γ_k', α_{jk}'. A computation shows that

$$\frac{1-|z|^2}{1-|\varphi(z)|^2} \to \frac{1}{2t - \sum_{j=2}^N |\alpha_{jk}|^2}$$

and

$$1-\rho^2(z) \to \frac{(2t - \sum_{j=2}^N |\alpha_{jk}|^2)(2t - \sum_{j=2}^N |\alpha_{jk}'|^2)}{|2t - \sum_{j=2}^N \alpha_{jk}\overline{\alpha_{jk}'}|^2}$$

as z tends to e_1 along the curve Γ_k. Similarly, using $t = d_\varphi(e_1) = d_\psi(e_1)$, we deduce that $\sum_{j=2}^N |\alpha_{jk}|^2 \leq t$ and $\sum_{j=2}^N |\alpha_{jk}'|^2 \leq t$. Let $L_1 = (\sum_{j=2}^N |\alpha_{jk}|^2)^{1/2}$, $L_2 = (\sum_{j=2}^N |\alpha_{jk}'|^2)^{1/2}$ and $I = \sum_{j=2}^N \alpha_{jk}\overline{\alpha_{jk}'}$. Since $C_\varphi - C_\psi$ is compact, by Theorem 3 of [13], we see that $1-\rho^2(z)$ must tend to 1 as $z \to e_1$ along Γ_k and so

$$\begin{aligned} 0 &= \left|2t - \sum_{j=2}^N \alpha_{jk}\overline{\alpha_{jk}'}\right|^2 - \left(2t - \sum_{j=2}^N |\alpha_{jk}|^2\right)\left(2t - \sum_{j=2}^N |\alpha_{jk}'|^2\right) \\ &= 2t(L_1^2 + L_2^2 - 2\mathrm{Re}\, I) + |I|^2 - L_1^2 L_2^2. \end{aligned}$$

The remaining proof is similar to that used in the proof of $\beta_j = \beta_j'$, so we omit it.

Finally, it remains to prove $\gamma_k = \gamma_k'$ for $2 \leq k \leq N$. Fix $k \geq 2$ and $0 < r < 1/2$, let $\Gamma_{k,r} \equiv \{z = z_1e_1 + (z_1 - 1)e_k \in B_N, z_1 = 1 - r + re^{i\theta} \text{ for real } \theta\}$. Then the curve $\Gamma_{k,r}$ approaches e_1 as $\theta \to 0$, i.e as $z \to e_1$ and for points $z \in \Gamma_{k,r}$, we have $\frac{1-|z_1|^2}{|1-z_1|^2} = \frac{1-r}{r}$, $\frac{1-|z|^2}{|1-z_1|^2} = \frac{1-2r}{r}$ and $\frac{1-Re\, z_1}{|1-z_1|^2} = \frac{1}{2r}$. As z tends to e_1 along $\Gamma_{k,r}$, we get that

$$\begin{aligned} \varphi_1(z) &= 1 + t(z_1-1) - t(K+\gamma_k)(z_1-1)^2 + o(|z_1-1|^2), \\ \varphi_j(z) &= -(\beta_j - \alpha_{jk})(z_1-1) + (K+\gamma_k)(\beta_j - \alpha_{jk})(z_1-1)^2 + o(|z_1-1|^2) \end{aligned}$$

for $j \geq 2$ and similar expansions for the components of ψ only with γ_k' replacing γ_k. Therefore,

$$\frac{1-|z|^2}{1-|\varphi(z)|^2} \to \frac{1-2r}{t - r[t^2 + 2t\mathrm{Re}\,(K+\gamma_k) + \sum_{j=2}^N |\beta_j - \alpha_{jk}|^2]}$$

as z approaches e_1 along $\Gamma_{k,r}$. By Theorem 3 in [13], this with the compactness of $C_\varphi - C_\psi$ shows that $\rho(z)$ converges to zero as $z \to e_1$ along $\Gamma_{k,r}$. It is clear that

$$\rho(z) \geq \left|\frac{\varphi(z) - \frac{<\psi(z),\varphi(z)>}{|\varphi(z)|^2}\varphi(z)}{1 - <\psi(z),\varphi(z)>}\right|$$

and

$$\left|\frac{\varphi-\frac{<\psi,\varphi>}{|\varphi|^2}\varphi}{1-<\psi,\varphi>}\right| = \frac{1}{|\varphi|}\left|\frac{<\varphi-\psi,\varphi>}{1-|\varphi|^2+<\varphi-\psi,\varphi>}\right|$$
$$= \frac{1}{|\varphi|}\left|\frac{-t(\gamma_k-\gamma_k')(z_1-1)^2+o(|z_1-1|^2)}{1-|\varphi|^2-t(\gamma_k-\gamma_k')(z_1-1)^2+o(|z_1-1|^2)}\right|$$
$$= \frac{1}{|\varphi|}\left|\frac{-t(\gamma_k-\gamma_k')\frac{(z_1-1)^2}{|z_1-1|^2}+o(1)}{\frac{1-|\varphi|^2}{|z_1-1|^2}-t(\gamma_k-\gamma_k')\frac{(z_1-1)^2}{|z_1-1|^2}+o(1)}\right|.$$

Taking the limit as $z\to e_1$ along $\Gamma_{k,r}$ and using $\frac{(z_1-1)^2}{|z_1-1|^2}\to -1$, the quotient converges to

$$\left|\frac{t(\gamma_k-\gamma_k')}{\frac{t}{r}-t^2-2t\mathrm{Re}\,K-t(\overline{\gamma_k}+\gamma_k')-\sum_{j=2}^N|\beta_j-\alpha_{jk}|^2}\right|.$$

It must be zero from the above discussions. So we have $\gamma_k=\gamma_k'$ as desired and complete the proof. □

Now, Combining the argument in Section 1 with the proof of Theorem B or as a corollary of Theorem 3.1, Theorem B can be improved as follows.

Theorem 3.2. *Suppose that φ is a linear fractional self-map of B_N with $||\varphi||_\infty=1$. The operator $C_{\varphi\circ\sigma}-C_{\sigma\circ\varphi}$ is compact on $H^2(B_N)$ or $A_s^2(B_N)$ $(s>-1)$ if and only if $\varphi\circ\sigma=\sigma\circ\varphi$.*

Proof. We only need to prove one direction. If $||\varphi||_\infty=1$, there exist ζ and η on ∂B_N such that $\varphi(\zeta)=\eta$, then $\sigma(\eta)=\zeta$ by Lemma 1 of [13]. This implies that $||\sigma\circ\varphi||_\infty=1$ for the linear fractional map $\sigma\circ\varphi$ of B_N and so $C_{\sigma\circ\varphi}$ is not compact. Therefore, if $C_{\varphi\circ\sigma}-C_{\sigma\circ\varphi}$ is compact, by Theorem 3.1, we have $\varphi\circ\sigma=\sigma\circ\varphi$. □

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